A METHOD OF CONSTRUCTING A SOLUTION OF THE HEAT-CONDUCTION EQUATION WITH COMPLEX BOUNDARY CONDITIONS

B. F. Ryzhenko

UDC 517.946:536.24

It is proposed to solve the heat conduction equation with complicated boundary conditions using the notion of R-functions. A solution which satisfies exactly mixed boundary conditions or boundary conditions of the first, second, or third kind is constructed.

It is generally not possible to solve the problem concerning the temperature distribution in an arbitrary three-dimensional domain owing to the complex form of the boundaries of such a domain.

Difficulties encountered in solving such a problem, which depends on a complicated boundary of the domain in question, may be circumvented by using the notion of R-functions [1, 2], which makes it possible to construct functions connected in a natural way with the form of the boundary of the domain and the conditions which must be satisfied on this boundary. The stationary temperature distribution in a domain $\bar{D} = D + \Gamma + \gamma$ may be obtained, as is well known, as a result of solving Poisson's equation

$$-\nabla^2\Theta(x, y, z) = q(x, y, z) \quad ((x, y, z)\in D) \tag{1}$$

with specified conditions on the unknown quantity $\Theta(x, y, z)$ on the boundary of this domain $\Gamma + \gamma$.

Let us assume that on $\Gamma + \gamma$ the temperature must satisfy the conditions

$$\left(\lambda \frac{\partial\Theta}{\partial n} + \alpha\Theta\right)\Big|_\Gamma = \varphi, \qquad \Theta|_\gamma = \psi. \tag{2}$$

Here λ, α, φ, ψ are piecewise-continuous differentiable functions of the coordinates of points belonging to the boundary.

The possibility always exists of decomposing the boundary Γ of the domain D into several elements $\Gamma_1, \Gamma_2, \ldots, \Gamma_m \left(\Gamma = \bigcup_{i=1}^{m}\Gamma_i\right)$, on each of which the function φ can be represented in the form of a single analytic expression. In this case the condition on the unknown function Θ must be satisfied on each element Γ_i:

$$\left(\lambda \frac{\partial\Theta}{\partial n} + \alpha\Theta\right)\Big|_{\Gamma_i} = [\lambda(\mathbf{n}, \nabla\Theta) + \alpha\Theta]|_{\Gamma_i} = \varphi_i \quad (i = 1, 2, \ldots, m). \tag{3}$$

Here, and henceforth, the subscript i indicates that the values of the corresponding functions are taken at points belonging to the element Γ_i.

Let $f_1(x, y, z) = 0$, $f_2(x, y, z) = 0, \ldots, f_m(x, y, z) = 0$ be the equations, respectively, of the boundary elements $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ of the domain being investigated. These equations may be formulated with the use of R-functions [3]. Following [1], we construct the functions

$$\Lambda = \sum_{i=1}^{m} \frac{f_1^2 \ldots f_{i-1}^2 f_{i+1}^2 \ldots f_m^2}{f_i^2 + f_1^2 \ldots f_{i-1}^2 f_{i+1}^2 \ldots f_m^2}\lambda_i, \tag{4}$$

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 21, No. 5, pp. 909-913, November, 1971. Original article submitted December 7, 1970.

$$A = \sum_{i=1}^{m} \frac{f_1^2 \ldots f_{i-1}^2 f_{i+1}^2 \ldots f_m^2}{f_i^2 + f_1^2 \ldots f_{i-1}^2 f_{i+1}^2 \ldots f_m^2} \alpha_i, \tag{5}$$

$$\Phi = \sum_{i=1}^{m} \frac{f_1^2 \ldots f_{i-1}^2 f_{i+1}^2 \ldots f_m^2}{f_i^2 + f_1^2 \ldots f_{i-1}^2 f_{i+1}^2 \ldots f_m^2} \varphi_i. \tag{6}$$

Since for $(x, y, z) \in \Gamma_i$ only $f_i = 0$, then

$$\Lambda = \lambda_i, \quad A = \alpha_i, \quad \Phi = \varphi_i \text{ on } \Gamma_i$$

and, consequently, the boundary conditions (3) may be written in the form

$$\left[\Lambda \left(\frac{\nabla\omega}{|\nabla\omega|}, \nabla\Theta \right) + A\Theta - \Phi \right] \Big|_{\Gamma} = 0. \tag{7}$$

Here $\omega(x, y, z)$ is a continuous function of real arguments, which has inside the domain D bounded and continuous derivatives $\partial\omega/\partial x$, $\partial\omega/\partial y$, $\partial\omega/\partial z$ and which satisfies in the domain D the condition $\omega(x, y, z) > 0$ and on the boundary $\Gamma + \gamma$ of this domain the condition $\omega(x, y, z) \equiv 0$. The vector $\nabla\omega/|\nabla\omega|$ has a length equal to one, and for $(x, y, z) \in \Gamma + \gamma$ it coincides with the unit normal vector $\Gamma + \gamma$.

Analogously, for the boundary γ we have:

1. Equations of the elements of this boundary

$$\chi_l(x, y, z) = 0 \quad (l = 1, 2, \ldots, n);$$

2.

$$\gamma = \bigcup_{l=1}^{n} \gamma_l; \tag{8}$$

3.

$$\Psi = \sum_{l=1}^{n} \frac{\chi_1^2 \ldots \chi_{l-1}^2 \chi_{l+1}^2 \ldots \chi_n^2}{\chi_l^2 + \chi_1^2 \ldots \chi_{l-1}^2 \chi_{l+1}^2 \ldots \chi_n^2} \psi_l.$$

4. The boundary condition for the unknown quantity

$$(\Theta - \Psi)|_{\gamma} = 0. \tag{9}$$

We seek a solution of the differential equation (1), satisfying the boundary conditions (7) and (9), in the form

$$\Theta(x, y, z) = \omega_1(x, y, z) F_1(x, y, z) + \omega_2(x, y, z) F_2(x, y, z) + F_3(x, y, z), \tag{10}$$

where F_1, F_2, F_3 are continuous, twice-differentiable functions, and the functions ω_1 and ω_2 satisfy the conditions

$$\omega_1(x, y, z)|_{\gamma} = 0,$$

$$\omega_2(x, y, z)|_{\Gamma+\gamma} = 0,$$

$$\omega_1(x, y, z) > 0, \quad \omega_2(x, y, z) > 0 \text{ in the domain D.}$$

In particular, we can put

$$\omega_1 = \chi_1^2 \chi_2^2 \ldots \chi_n^2, \quad \omega_2 \equiv \omega.$$

Substituting the expression (10) into Eq. (9), we find that this boundary condition for Θ is satisfied if $F_3 \equiv \Psi$. Using the boundary condition (7), we obtain a relationship connecting the values of the functions F_1 and F_2

$$\left[\Lambda \left(\frac{\nabla\omega}{|\nabla\omega|}, \nabla(\omega_1 F_1) \right) + \Lambda \left(\frac{\nabla\omega}{|\nabla\omega|}, \nabla\omega_2 \right) F_2 + \Lambda \left(\frac{\nabla\omega}{|\nabla\omega|}, \nabla\Psi \right) + A\omega_1 F_1 + A\Psi - \Phi \right] \Big|_{\Gamma} = 0.$$

From this it follows that the function

$$\Theta = (\Psi + \omega_1 F_1) + \omega_2 \frac{\Phi - A(\Psi + \omega_1 F_1) - \Lambda\left(\frac{\nabla\omega}{|\nabla\omega|}, \nabla(\Psi + \omega_1 F_1) \right)}{\Lambda \left(\frac{\nabla\omega}{|\nabla\omega|}, \nabla\omega_2 \right)} \tag{11}$$

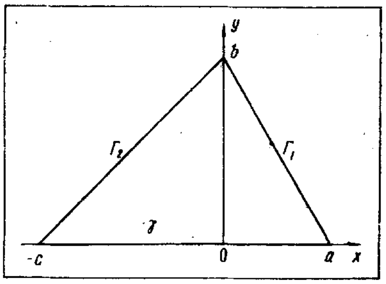

Fig. 1. Region in which the temperature field is to be determined.

satisfies the given boundary conditions (7) and (9) independently of the choice of the function $F_1(x, y, z)$. The function $F_1(x, y, z)$ may be chosen arbitrarily to satisfy approximately the requirements of the solution constructed for Poisson's equation. Putting

$$F_1(x, y, z) = \sum_{j=0}^{N} c_j g_j(x, y, z), \tag{12}$$

where $\{g_j(x, y, z)\}$ is a complete system of functions, we can determine the coefficients c_j by one of the variational methods, for example, by using the functional [4]

$$I(\Theta) = \int_D [(\nabla\Theta)^2 - 2\Theta q]\, d\Omega. \tag{13}$$

Having done this, we will have solved the problem of determining the temperature distribution in a region of three-dimensional space.

From Eq. (11) it follows that if on the boundary of the domain considered a boundary condition only of the first kind is given, then the solution of the problem satisfying this condition is sought in the form

$$\Theta = \Psi + \omega_1 F_1. \tag{14}$$

In the case when the unknown quantity on the boundary of the domain satisfies a condition of the second or third kind, we seek the solution satisfying this condition in the form

$$\Theta = F_1 + \omega \frac{\Phi - AF_1 - \Lambda\left(\frac{\nabla\omega}{|\nabla\omega|}, \nabla F_1\right)}{\Lambda \frac{(\nabla\omega)^2}{|\nabla\omega|}}. \tag{15}$$

To illustrate the method we construct a function describing the temperature field in a two-dimensional triangular domain (see Fig. 1) on one of the boundaries of which the temperature distribution is given and on the other two of which heat transfer takes place:

$$\Theta|_\gamma = \psi(x), \tag{16}$$

$$\left(\lambda \frac{\partial\Theta}{\partial n} + \alpha\Theta\right)\Big|_{\Gamma_1\cup\Gamma_2} = 0. \tag{17}$$

Using the R-conjunction (the ordinary one and its modification), we find, in accord with the theorems given in [3], that

$$\omega(x, y) \equiv \left[\frac{y}{b}\left(1 - \frac{y}{b}\right)\right] \Lambda_1 \left(\left[1 - \frac{x}{a} - \frac{y}{b}\right] \Lambda_1 \left[1 + \frac{x}{c} - \frac{y}{b}\right]\right),$$

$$f_1(x, y) \equiv \left[\frac{y}{b}\left(1 - \frac{y}{b}\right)\right] \Lambda_2 \left[1 - \frac{x}{a} - \frac{y}{b}\right],$$

$$f_2(x, y) \equiv \left[\frac{y}{b}\left(1 - \frac{y}{b}\right)\right] \Lambda_2 \left[1 + \frac{x}{c} - \frac{y}{b}\right],$$

$$\chi(x, y) \equiv \left[\left(1 - \frac{x}{a}\right)\left(1 + \frac{x}{c}\right)\right] \Lambda_2 \frac{y}{b},$$

where $X \Lambda_1 Y = (X + Y - \sqrt{X^2 + Y^2})/2$ is the R-conjunction and $X \Lambda_2 Y = X - \sqrt{X^2 + Y^2}$ is the modified R-conjunction. Then

$$\Lambda = \lambda, \quad A = \alpha, \quad \Phi \equiv 0, \quad \Psi = \psi(x)$$

and the solution of the given problem may be written in the form ($\omega_1 = \chi^2$, $\omega_2 \equiv \omega$)

$$\Theta = (\psi + \chi^2 F) - \omega \frac{\alpha(\psi + \chi^2 F) + \lambda\left(\frac{\nabla\omega}{|\nabla\omega|}, \ \nabla(\psi + \chi^2 F)\right)}{\lambda \frac{(\nabla\omega)^2}{|\nabla\omega|}}. \tag{18}$$

The unknown function $F \equiv F(x, y)$ may be approximated by the polynomial

$$F(x, y) = \sum_{i+j=0}^{N} c_{ij} x^i y^j. \tag{19}$$

NOTATION

$\Theta(x, y, z)$	is the temperature distribution function;
$q(x, y, z)$	is the function of heat-source distribution in the space region under consideration;
$\lambda(\Lambda)$	is the thermal conductivity;
$\alpha(A)$	is the heat-transfer coefficient;
$\varphi(\Phi)$ and $\psi(\Psi)$	are the values of the unknown temperature at the boundary of the region;
ω, ω_1, ω_2, f_i, χ_l	arethe functions with constant sign in the region and turning to zero only at the boundary or its separate sections;
$\gamma(\gamma_l)$, $\Gamma(\Gamma_i)$	are the sections of the boundary in the region D under study.

LITERATURE CITED

1. V.L. Rvachev, Geometrical Applications of Algebraic Logic [in Russian], Tekhnika, Kiev (1967).
2. V. L. Rvachev and K. L. Yushchenko, Dokl. Akad. Nauk UkrSSR, No.2, 163-166 (1965).
3. V. L. Rvachev, Dokl. Akad. Nauk SSSR, 153, No. 4 (1963).
4. S. G. Mikhlin, Direct Methods in Mathematical Physics [in Russian], GITTL, Moscow–Leningrad (1950).